Chapter 6 Reflection and Transmission of Waves

6.1 Boundary Conditions

At the boundary of two different medium, electromagnetic field have to satisfy physical condition, which is determined by Maxwell's equation. This is the boundary condition to be applied to any electromagnetic field.

$$\hat{\boldsymbol{n}} \times (\boldsymbol{H_1} - \boldsymbol{H_2}) = \boldsymbol{J_S} \tag{6.1.1}$$

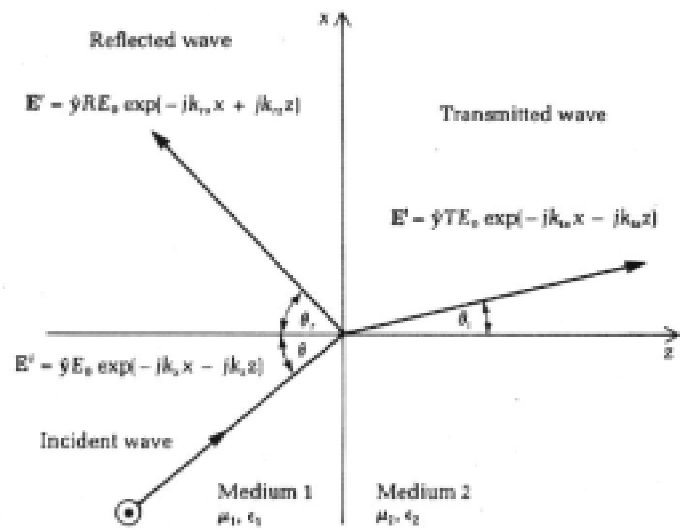
$$\hat{\boldsymbol{n}} \times (\boldsymbol{E_1} - \boldsymbol{E_2}) = 0 \tag{6.1.2}$$

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{D_1} - \boldsymbol{D_2}) = \rho_S \tag{6.1.3}$$

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{B_1} - \boldsymbol{B_2}) = 0 \tag{6.1.4}$$

From the Maxwell's equation, we can find these four boundary conditions. Here $\hat{\boldsymbol{n}}$ denotes the unit normal vector to the boundary surface. Physical meanings of these equations can be described as follows. The tangential electric field $\boldsymbol{E}$ is continuous across the boundary surface. The discontinuity in the tangential magnetic field $\boldsymbol{H}$ is equal to the surface current $\boldsymbol{J_S}$. The normal component $\boldsymbol{B}$ is continuous across the boundary surface. The discontinuity in the normal component of $\boldsymbol{D}$ is equal to the surface charge density ρ_S.

6.2 Reflection and Transmission at a Dielectric Interface



In section 5.2, we descried a plane wave in two-dimensional space. Now we consider a plane wave impinges upon a plane dielectric interface, as shown in Figure. When the incident wave impinges on the boundary at an oblique angle, the normal of the boundary and the incident ray form a plane called the plane of incidence. The $\boldsymbol{E}$ field of the incident wave may be polarized perpendicular or parallel to the plane of incidence.

We now consider a perpendicularly polarized incident wave. The incident wave can be expressed as:

$$\boldsymbol{E}^i = \hat{\boldsymbol{y}} E_0 e^{-jk_x x - jk_z z} \tag{6.2.1}$$

$$\boldsymbol{H}^i = (-\hat{\boldsymbol{x}} k_z + \hat{\boldsymbol{z}} k_x) \frac{E_0}{\omega \mu_1} e^{-jk_x x - jk_z z} \tag{6.2.2}$$

The reflected wave is given by

$$\boldsymbol{E}^r = \hat{\boldsymbol{y}} R_I E_0 e^{-jk_{rx}x + jk_{rz}z} \tag{6.2.3}$$

$$\boldsymbol{H}^r = (+\hat{\boldsymbol{x}} k_{rz} + \hat{\boldsymbol{z}} k_{rx}) \frac{R_I E_0}{\omega\mu_1} e^{-jk_{rx}x + jk_{rz}z} \tag{6.2.4}$$

where R_I is the reflection coefficient for the wave. The wave vector of the reflected wave is

$$\boldsymbol{k_r} = \hat{\boldsymbol{x}} k_{rx} - \hat{\boldsymbol{z}} k_{rz} \tag{6.2.5}$$

And the transmitted wave is give by

$$\boldsymbol{E}^t = \hat{\boldsymbol{y}} T_I E_0 e^{-jk_{tx}x - jk_{tz}z} \tag{6.2.6}$$

$$\boldsymbol{H}^t = (-\hat{\boldsymbol{x}} k_{tz} + \hat{\boldsymbol{z}} k_{tx}) \frac{T_I E_0}{\omega\mu_2} e^{-jk_{tx}x - jk_{tz}z} \tag{6.2.7}$$

where T_I is the transmission coefficient. When neither of two media is a perfect conductor, the surface current $\boldsymbol{J_S} = 0$. Then, the boundary conditions (6.1.1) and (6.1.2) require that both the tangential electric field and magnetic field components be continuous at $z = 0$. We thus have

$$e^{-jk_x x} + R_I e^{-jk_x x} = T_I e^{-jk_{tx} x} \tag{6.2.8}$$

$$-\frac{k_z}{\omega\mu_1} e^{-jk_x x} + \frac{k_{rz}}{\omega\mu_1} R_I e^{-jk_{rx} x} = -\frac{k_{tz}}{\omega\mu_2} T_I e^{-jk_{tx} x} \tag{6.2.9}$$

these equations have to be satisfied for all values of x. The consequence is that

$$k_x = k_{rx} = k_{tx} \tag{6.2.10}$$

It means that the tangential component of the three wave vectors $\boldsymbol{k}$, $\boldsymbol{k_r}$, and $\boldsymbol{k_t}$ are equal. This condition is known as the phase matching condition.

We can obtain the magnitude of the three wave vectors by substituting the solution for $\boldsymbol{E^i}$, $\boldsymbol{E}^r$ and $\boldsymbol{E}^t$ into the wave equation

$$(\Delta + \omega^2 \mu_1 \varepsilon_1) \begin{Bmatrix} \boldsymbol{E}^i \\ \boldsymbol{E}^r \end{Bmatrix} = 0 \tag{6.2.11}$$

$$(\Delta + \omega^2 \mu_2 \varepsilon_2) \boldsymbol{E^t} = 0 \tag{6.2.12}$$

We find

$$k_x^2 + k_z^2 = \omega^2 \mu_1 \varepsilon_1 = k_1^2 \tag{6.2.13}$$

$$k_{rx}^2 + k_{rz}^2 = \omega^2 \mu_1 \varepsilon_1 = k_1^2 \qquad (6.2.14)$$

and

$$k_{tx}^2 + k_{tz}^2 = \omega^2 \mu_2 \varepsilon_2 = k_2^2 \qquad (6.2.15)$$

From the phase matching condition we find $k_x = k_{rx} = k_{tx}$ and $k_z = k_{rz}$. Using these result in (6.2.8) and (6.2.9) we obtain

$$1 + R_I = T_I \qquad (6.2.16)$$

$$1 - R_I = -\frac{\mu_1 k_{tz}}{\mu_2 k_z} T_I \qquad (6.2.17)$$

and solving these equations for R_I and T_I gives:

$$R_I = \frac{\mu_2 k_z - \mu_1 k_{tz}}{\mu_2 k_z + \mu_1 k_{tz}} \qquad (6.2.18)$$

$$T_I = \frac{2\mu_2 k_z}{\mu_2 k_z + \mu_1 k_{tz}} \qquad (6.2.19)$$

Referring to the angle of incidence θ and (5.2.2),

$$k_x = k_1 \sin\theta \qquad (6.2.20a)$$

$$k_{rx} = k_1 \sin\theta_r \qquad (6.2.20b)$$

$$k_{tx} = k_2 \sin\theta_t \qquad (6.2.20c)$$

Substituting these equations into the phase matching condition, (6.2.10), we find

$$k_1 \sin\theta_r = k_1 \sin\theta = k_2 \sin\theta_t \qquad (6.2.21)$$

The first equal sign states that $\theta_r = \theta$, that is the angle of reflection is equal to the angle of incidence. By using the definition used in optics, refractive indices

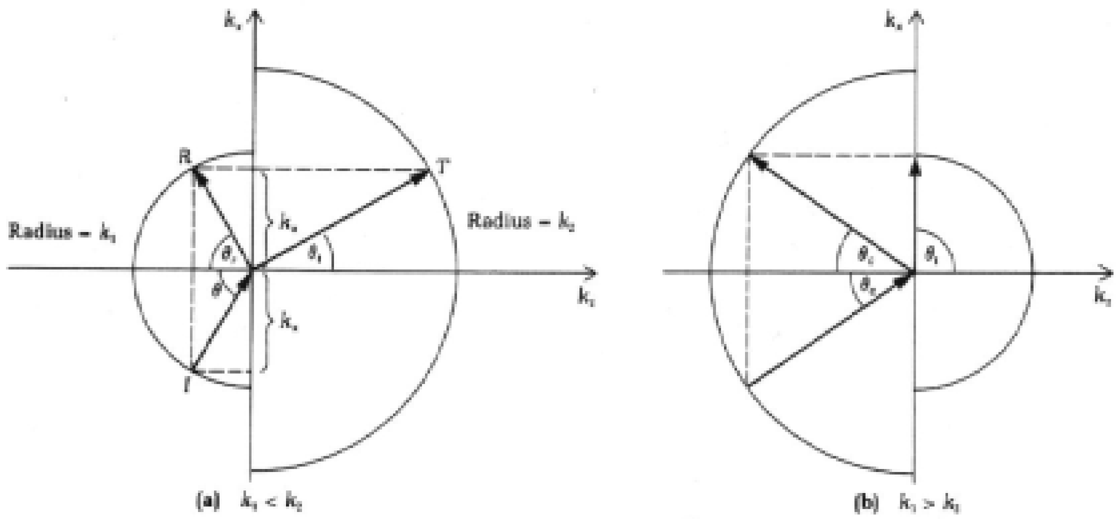
$$n_1 = c\sqrt{\mu_1 \varepsilon_1} = \frac{c}{\omega} k_1 \qquad (6.2.22a)$$

$$n_2 = c\sqrt{\mu_2 \varepsilon_2} = \frac{c}{\omega} k_2 \qquad (6.2.22b)$$

the phase matching condition $k_{tx} = k_x$ gives rise to

$$n_1 \sin\theta = n_2 \sin\theta_t \qquad (6.2.23)$$

this is the Snell's law.



The phase matching condition can be represented graphically. When $n_1 < n_2$, we can find vectors $\boldsymbol{k_r}$, and $\boldsymbol{k_t}$ for the given $\boldsymbol{k}$. However, when $n_1 > n_2$, for the angle greater than θ_c, k_x is larger than the magnitude of k_2. In that case,

$$k_{tz}^2 = k_2^2 - k_x^2 < 0 \tag{6.2.24}$$

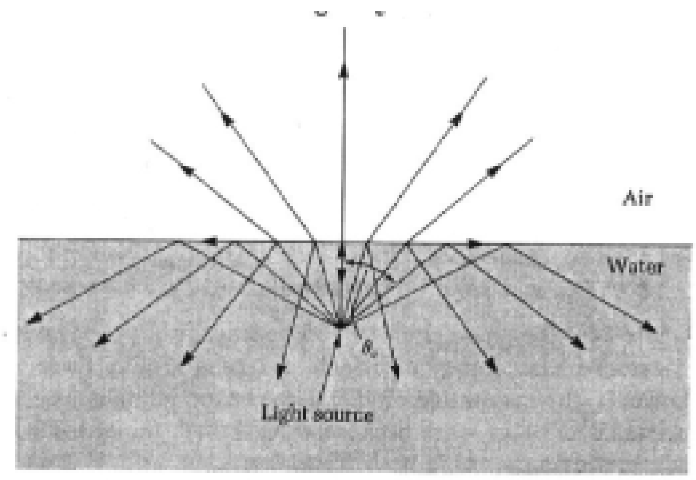
or

$$k_{tz} = \pm j\alpha \tag{6.2.25}$$

with $\alpha = \sqrt{k_x^2 - k_2^2}$ being a positive real number. In this case, the wave attenuates exponentially in the $+\hat{\boldsymbol{z}}$ direction. The transmitted electric field can be given by

$$\boldsymbol{E^t} = \hat{\boldsymbol{y}} T E_0 e^{-\alpha z} e^{-jk_x x} \tag{6.6.26}$$

which represents a no uniform plane wave or a surface wave. For example, the permittivity of the water at optical frequency is $1.77\varepsilon_0$ and the critical angle is given by $\theta_c = \sin^{-1}(\frac{1}{\sqrt{1.77}}) = 49°$.



An incident wave of arbitrary polarization can be decomposed into two waves having perpendicular and parallel polarizations. The electric filed of the perpendicularly polarized wave is perpendicular to the plane of incidence, and the parallel polarized wave's electric field is parallel to that plane. We now consider a parallel polarized incident wave. The situation of electric and magnetic field is shown in the figure and thy can be described by

$$\boldsymbol{H}^i = \hat{\boldsymbol{y}} H_0 e^{-jk_x x - jk_z z} \tag{6.2.27}$$

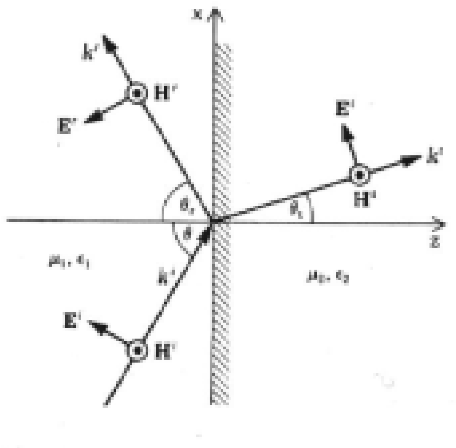
$$\boldsymbol{E}^i = (\hat{\boldsymbol{x}} k_z - \hat{\boldsymbol{z}} k_x) \frac{H_0}{\omega\varepsilon_1} e^{-jk_x x - jk_z z} \tag{6.2.28}$$

$$\boldsymbol{H}^r = \hat{\boldsymbol{y}} R_{II} H_0 e^{-jk_{rx} x + jk_{rz} z} \tag{6.2.29}$$

$$\boldsymbol{E}^r = (-\hat{\boldsymbol{x}} k_{rz} - \hat{\boldsymbol{z}} k_{rx}) \frac{R_{II} H_0}{\omega\varepsilon_1} e^{-jk_{rx} x + jk_{rz} z} \tag{6.2.30}$$

$$\boldsymbol{H}^t = \hat{\boldsymbol{y}} T_{II} H_0 e^{-jk_{tx} x - jk_{tz} z} \tag{6.2.31}$$

$$\boldsymbol{E}^t = (\hat{\boldsymbol{x}} k_{tz} - \hat{\boldsymbol{z}} k_{tx}) \frac{T_{II} H_0}{\omega\varepsilon_2} e^{-jk_{tx} x - jk_{tz} z} \tag{6.2.32}$$



Where R_{II} and T_{II} are , respectively, the reflection and transmission coefficients for the magnetic field vector for the parallelly polarized wave. They can be given by applying the boundary condition to (6.2.27) to (6.2.32) as:

$$T_{II} = \frac{2\varepsilon_2 k_z}{\varepsilon_2 k_z + \varepsilon_1 k_{tz}} \tag{6.2.33}$$

$$R_{II} = \frac{\varepsilon_2 k_z - \varepsilon_1 k_{tz}}{\varepsilon_2 k_z + \varepsilon_1 k_{tz}} \tag{6.2.34}$$

From (6.2.33), when $\mu_1 = \mu_2$, $R_{II} = 0$ gives

$$\omega\sqrt{\mu_1\varepsilon_2}\cos\theta_b = \omega\sqrt{\mu_1\varepsilon_1}\cos\theta_t \tag{6.2.35}$$

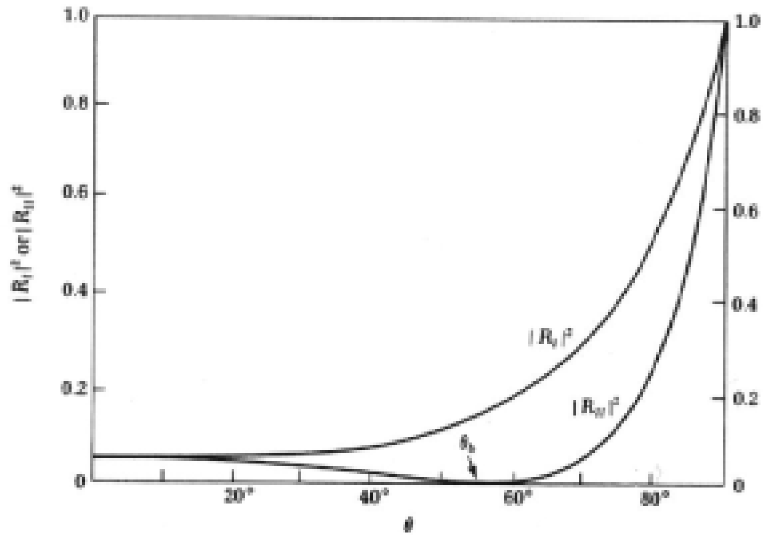
and the phase matching condition gives

$$\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_b = \omega\sqrt{\mu_1\varepsilon_2}\sin\theta_t \tag{6.2.36}$$

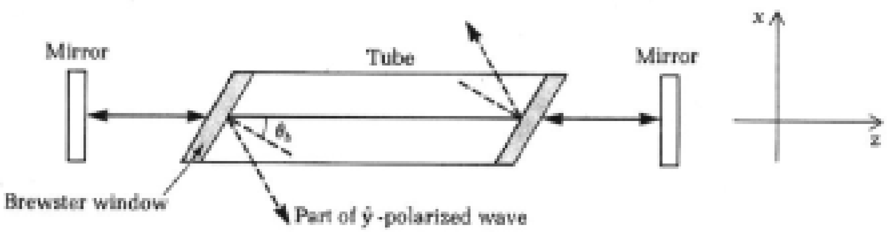
Solving the above two equations, we find $\theta_t + \theta_b = \dfrac{\pi}{2}$ and

$$\theta_b = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \tag{6.2.37}$$

where the incident angle θ_b is called Brewster angle.



Reflection power as a function of incident angle.
The martial is glass with $\varepsilon = 2.25\varepsilon_0$. The Brewster angle is $56°$.



A gas laser with Brewster window

6.3 Standing Waves

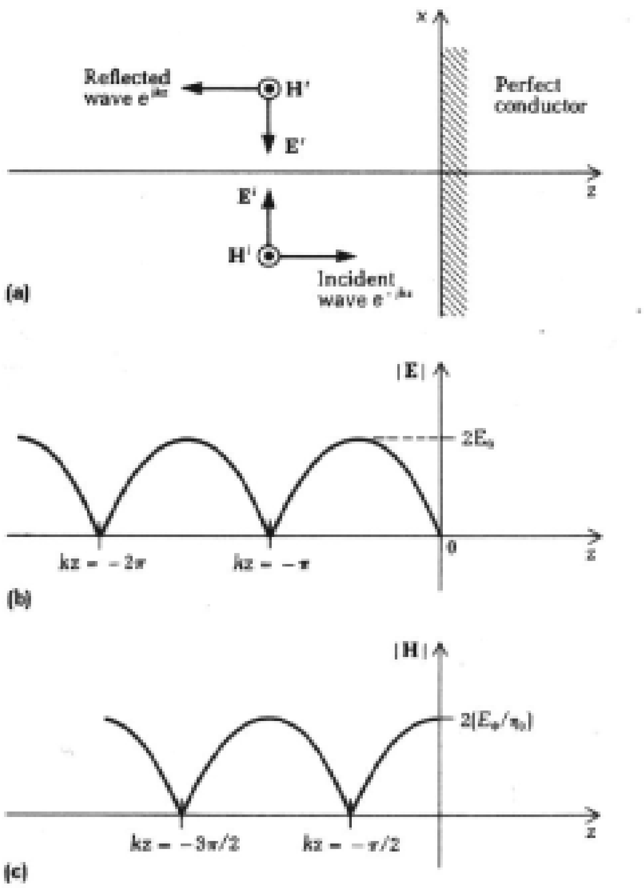
The complex permittivity is defined as (4.6.4) and is given by

$$\varepsilon = \varepsilon - j\frac{\sigma}{\omega} \qquad (6.3.1)$$

The perfect conductor is a medium with infinite conductivity. And from (6.3.1) we find that the perfect conductor can be regarded as a medium with infinite permittivity. Substituting $\varepsilon_2 \to \infty$ and $k_{tz} \approx \omega\sqrt{\mu\varepsilon_2} \to \infty$ we obtain

$$R_I = -1 \qquad (6.3.2)$$
$$R_{II} = 1 \qquad (6.3.3)$$

Consider a perfectly conducting half-space as shown in figure. A uniform plane wave impinges normally on the boundary is given by



$$\boldsymbol{E^i} = \hat{\boldsymbol{x}}E_0 e^{-jkz} \qquad (6.3.4)$$

$$\boldsymbol{H^i} = \hat{\boldsymbol{y}}\left(\frac{E_0}{\eta_0}\right)e^{-jkz} \qquad (6.3.5)$$

And the reflected wave is given by:

$$\boldsymbol{E}^r = -\hat{\boldsymbol{x}}E_0 e^{jkz} \qquad (6.3.6)$$

$$\boldsymbol{H^r} = \hat{\boldsymbol{y}}\left(\frac{E_0}{\eta_0}\right)e^{jkz} \qquad (6.3.7)$$

The total electromagnetic field in medium 1 is the sum of the incident and the reelected waves

$$\boldsymbol{E} = \hat{\boldsymbol{x}}E_0(e^{-jkz} - e^{jkz}) = -\hat{\boldsymbol{x}}2jE_0 \sin kz \qquad (6.3.8)$$

$$\boldsymbol{H} = \hat{\boldsymbol{y}}\frac{E_0}{\eta_0}(e^{-jkz} + e^{jkz}) = \hat{\boldsymbol{y}}\left(\frac{2E_0}{\eta_0}\right)\cos kz \qquad (6.3.9)$$

The boundary condition (6.1.1) sates at *z=0* gives the surface current on the perfect conductor.

$$\boldsymbol{J}_S = (-\hat{\boldsymbol{z}}) \times \boldsymbol{H} = \hat{\boldsymbol{x}}\frac{2E_0}{\eta_0} \qquad (6.3.10)$$

The instantaneous values of electric ad magnetic field are given by:

$$\boldsymbol{E} = \hat{\boldsymbol{x}} 2E_0 \sin kz \sin \omega t \qquad (6.3.11)$$

$$\boldsymbol{H} = \hat{\boldsymbol{y}} 2\left(\frac{E_0}{\eta_0}\right) \cos kz \cos \omega t \qquad (6.3.12)$$

and they are plotted in the figure. These patters are called standing-wave patters because the waveform does not shift in space as time processes.

6.4 Standing Wave in front of a Dielectric Medium

Consider a uniform plane wave impinges on a plane dielectric interface. The total $\boldsymbol{E}$ field in medium 1 is given as:

$$\boldsymbol{E}_1 = \hat{\boldsymbol{y}} E_0 (e^{-jk_1 z} + R_I e^{jk_1 z}) \qquad (6.4.1)$$

and the $\boldsymbol{E}$ field in medium 2 is

$$\boldsymbol{E}_2 = \hat{\boldsymbol{y}} T_I E_0 e^{-jk_2 z} \qquad (6.4.2)$$

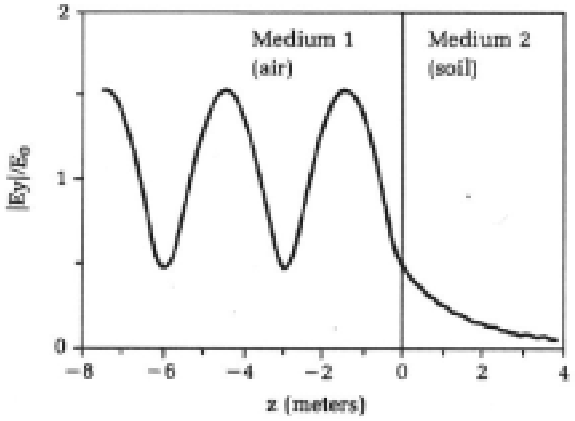
The reflection and the transmission coefficients are given by (6.2.18) and (6.2.19). Assume that medium 1 is air and medium 2 is soil characterized by $\varepsilon = 10\varepsilon_0$, $\sigma = 0.01\, S/m$, and $\mu = \mu_0$. When the frequency is 50MHz, we have:

$$k_z = k \cos\theta = k_1 = 1.048 \qquad (6.4.3)$$

$$k_{1z} = \sqrt{k_2^2 - k_x^2} = k_2 = 3.365 - j0.587 \qquad (6.4.4)$$

$$R_I = 0.537 e^{j173.4^\circ} \qquad (6.4.5)$$

$$T_I = 0.471 e^{-j0.587z} \qquad (6.4.6)$$



Substituting the R_I value in (6.4.2) we have

$$\left|E_{1y}\right| = E_0 \left|1 + 0.537 e^{+j(2k_{1z} + 173.4^\circ)}\right| \qquad (6.4.7)$$

and similarly,

$$\left|E_{2y}\right| = 0.471 E_0 e^{-0.587z} \qquad (6.4.8)$$